SOME RESEARCH PROBLEMS ON THE PERMUTATION BEHAVIOUR OF REVERSED DICKSON POLYNOMIALS

NERANGA FERNANDO

Find the conditions on k and l explicitly for which $D_{p^l+3,k}(1,x)$ is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.1. *Assume that*

(1) $p=5$ *and* $k\neq 2$,
(2) $p>5$ *and* $k\neq 0,2$, *or*
(3) $p>7$ *and* $k\neq 7$.

Then $D_{p^l+3,k}(1,x)$ *is a PP of* $\mathbb{F}_{p^e}$ *if and only if*

$$(2-k)\,x^{\frac{p^l+3}{2}}+6\,x^{\frac{p^l+1}{2}}+k\,x^{\frac{p^l-1}{2}}+2(3-k)\,x$$

is a PP of $\mathbb{F}_{p^e}$.

Find the conditions on k and l_i's explicitly for which $D_{n,k}(1,x)$ is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.2. *Let* $k=1$, $p=3$, *and* $n=p^{l_1}+p^{l_2}+p^{l_3}$. *Assume that* $l_1\neq 0$, $l_2\neq 0$ *and* $l_3\neq 0$. *Then* $D_{n,k}(1,x)$ *is a PP of* $\mathbb{F}_{p^e}$ *if and only if*

$$g(x)=x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}-1}{2}}+x^{\frac{p^{l_1}+p^{l_2}}{2}}+x^{\frac{p^{l_1}+p^{l_3}}{2}}+x^{\frac{p^{l_2}+p^{l_3}}{2}}+x^{\frac{p^{l_1}-1}{2}}+x^{\frac{p^{l_2}-1}{2}}+x^{\frac{p^{l_3}-1}{2}}.$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.3. *Let* $k=2$, $p=3$, *and* $n=p^{l_1}+p^{l_2}+p^{l_3}$. *Assume that* $l_1\neq 0$, $l_2\neq 0$ *and* $l_3\neq 0$. *Then* $D_{n,k}(1,x)$ *is a PP of* $\mathbb{F}_{p^e}$ *if and only if*

$$g(x)=x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}-1}{2}}+x^{\frac{p^{l_1}-1}{2}}+x^{\frac{p^{l_2}-1}{2}}+x^{\frac{p^{l_3}-1}{2}}.$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.4. *Let* $p>3$, $k\neq 3$, *and* $\frac{3k}{2(k-3)}$ *be a quadratic non-residue of* p. *Then* $D_{p^{l_1}+p^{l_2}+p^{l_3},k}(1,x)$ *is a PP of* $\mathbb{F}_{p^e}$ *if and only if*

$$k\,x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}-1}{2}}+(2-k)\,[x^{\frac{p^{l_1}+p^{l_2}}{2}}+x^{\frac{p^{l_1}+p^{l_3}}{2}}+x^{\frac{p^{l_2}+p^{l_3}}{2}}]+k[x^{\frac{p^{l_1}-1}{2}}+x^{\frac{p^{l_2}-1}{2}}+x^{\frac{p^{l_3}-1}{2}}].$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.5. *Let* $k=0$, $p=3$, *and* $n=p^{l_1}+p^{l_2}+p^{l_3}+p^{l_4}$. *Assume that exactly two of* l_1,l_2,l_3, *and* l_4 *are zero and the two non zero* l_i *have different parity. Then* $D_{n,k}(1,x)$ *is a PP of* $\mathbb{F}_{p^e}$ *if and only if*

$$g(x)=x^{\frac{p^{l_1}+p^{l_2}}{2}+1}+x^{\frac{p^{l_1}+p^{l_2}}{2}}+2x^{\frac{p^{l_1}+1}{2}}+2x^{\frac{p^{l_2}+1}{2}}+x$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.6. *Let $k = 1$, $p = 3$, and $n = p^{l_1} + p^{l_2} + p^{l_3} + p^{l_4}$. Assume that exactly two of l_1, l_2, l_3, and l_4 are zero and the two non-zero l_i are both even. Then $D_{n,k}(1, x)$ is a PP of $\mathbb{F}_{p^e}$ if and only if*

$$g(x) = x^{\frac{p^{l_1}+p^{l_2}}{2}+1} + x^{\frac{p^{l_1}-1}{2}} + x^{\frac{p^{l_2}-1}{2}} + x$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.7. *Let $k = 0$, $p = 3$, and $n = p^{l_1} + p^{l_2} + p^{l_3} + p^{l_4}$. Assume that exactly one of l_1, l_2, l_3, and l_4 is zero. Assume that non-zero l_i are all odd or exactly one of them is odd. Then $D_{n,k}(1, x)$ is a PP of $\mathbb{F}_{p^e}$ if and only if*

$$g(x) = x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}+1}{2}} + x^{\frac{p^{l_1}+p^{l_2}}{2}} + x^{\frac{p^{l_1}+p^{l_3}}{2}} + x^{\frac{p^{l_2}+p^{l_3}}{2}} + x^{\frac{p^{l_1}+1}{2}} + x^{\frac{p^{l_2}+1}{2}} + x^{\frac{p^{l_3}+1}{2}}$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.8. *Let $k = 0$, $p = 3$, and $n = p^{l_1} + p^{l_2} + p^{l_3} + p^{l_4}$. Assume that $l_1 \neq 0, l_2 \neq 0, l_3 \neq 0$, and $l_4 \neq 0$. Further assume that either exactly one of l_1, l_2, l_3 and l_4 is odd or exactly three of l_1, l_2, l_3 and l_4 are odd. Then $D_{n,k}(1, x)$ is a PP of $\mathbb{F}_{p^e}$ if and only if*

$$g(x) = x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}+p^{l_4}}{2}} + x^{\frac{p^{l_1}+p^{l_2}}{2}} + x^{\frac{p^{l_1}+p^{l_3}}{2}} + x^{\frac{p^{l_2}+p^{l_3}}{2}} + x^{\frac{p^{l_1}+p^{l_4}}{2}} + x^{\frac{p^{l_2}+p^{l_4}}{2}} + x^{\frac{p^{l_3}+p^{l_4}}{2}}.$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.9. *Let $p \geq 5$, $0 \leq k \leq p-1$ with $k \neq 2$. Assume that $\frac{2(k-6)}{(2-k)}$ is a quadratic non-residue of p and at least two of l_1, l_2, l_2, and l_4 are non-zero. Then $D_{n,k}(1, x)$ is a PP of $\mathbb{F}_{p^e}$ if and only if*

$$\begin{aligned}&(2-k)\,x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}+p^{l_4}}{2}} + k\,[x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}-1}{2}} + x^{\frac{p^{l_1}+p^{l_2}+p^{l_4}-1}{2}} + x^{\frac{p^{l_1}+p^{l_3}+p^{l_4}-1}{2}} + x^{\frac{p^{l_2}+p^{l_3}+p^{l_4}-1}{2}}]\\ &+ (2-k)\,[x^{\frac{p^{l_1}+p^{l_2}}{2}} + x^{\frac{p^{l_1}+p^{l_3}}{2}} + x^{\frac{p^{l_2}+p^{l_3}}{2}} + x^{\frac{p^{l_1}+p^{l_4}}{2}} + x^{\frac{p^{l_2}+p^{l_4}}{2}} + x^{\frac{p^{l_3}+p^{l_4}}{2}}]\\ &+ k\,[x^{\frac{p^{l_1}-1}{2}} + x^{\frac{p^{l_2}-1}{2}} + x^{\frac{p^{l_3}-1}{2}} + x^{\frac{p^{l_4}-1}{2}}].\end{aligned}$$

is a PP of $\mathbb{F}_{p^e}$.

Theorem 0.10. *Let $p \equiv 3 \pmod 4$ and $k = 2$. Then $D_{p^{l_1}+p^{l_2}+p^{l_3}+p^{l_4},k}(1, x)$ is a PP of $\mathbb{F}_{p^e}$ if and only if*

$$h(x) = x^{\frac{p^{l_1}+p^{l_2}+p^{l_3}-1}{2}} + x^{\frac{p^{l_1}+p^{l_2}+p^{l_4}-1}{2}} + x^{\frac{p^{l_1}+p^{l_3}+p^{l_4}-1}{2}} + x^{\frac{p^{l_2}+p^{l_3}+p^{l_4}-1}{2}} + x^{\frac{p^{l_1}-1}{2}} + x^{\frac{p^{l_2}-1}{2}} + x^{\frac{p^{l_3}-1}{2}} + x^{\frac{p^{l_4}-1}{2}}$$

is a PP of of $\mathbb{F}_{p^e}$.

0.1. Permutation behaviour of $D_{p^{l_1}+p^{l_2},k}$.

$$D_{p^{l_1}+p^{l_2},k}(1, x) = \frac{(2-k)}{4}\,(1-4x)^{\frac{p^{l_1}+p^{l_2}}{2}} + \frac{k}{4}\,(1-4x)^{\frac{p^{l_1}-1}{2}} + \frac{k}{4}\,(1-4x)^{\frac{p^{l_2}-1}{2}} + \frac{(2-k)}{4}.$$

Corollary 0.11. *Let $p = 3$ and $k = 2$. Assume that both l_1 and l_2 are odd. Then $D_{p^{l_1}+p^{l_2},k}(1, x)$ is a PP of $\mathbb{F}_{p^e}$ if and only if the binomial $x^{\frac{p^{l_1}-1}{2}} + x^{\frac{p^{l_2}-1}{2}}$ is a PP of $\mathbb{F}_q$.*

Corollary 0.12. *Let $k \neq 0, 2$ and $p > 3$. Assume that $\frac{2k}{(k-2)}$ is a quadratic non-residue of p. Then $D_{p^{l_1}+p^{l_2},k}(1, x)$ is a PP of $\mathbb{F}_{p^e}$ if and only if the trinomial $(2-k)\,x^{\frac{p^{l_1}+p^{l_2}}{2}} + k\,x^{\frac{p^{l_1}-1}{2}} + k\,x^{\frac{p^{l_2}-1}{2}}$ is a PP of $\mathbb{F}_{p^e}$.*

References

[1] N. Fernando, *Reversed Dickson polynomials of the* $(k+1)$*-th kind over finite fields, II*, arXiv:1706.01391

Department of Mathematics, Northeastern University, Boston, MA 02115
E-mail address: w.fernando@northeastern.edu